

# Inhomogenous Poisson Networks and Random Cellular Structures

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## Abstract

we study the statistical properties of inhomogenous Poisson networks. we perform a detailed analysis of the statistical properties of Poisson networks and show that the topological properties of random cellular structures, can be derived from these models of random networks. we study both two and three dimensional networks with non uniform density and show that with a class of symmetric distribution  $P(\lambda_1, \lambda_2, \dots, \lambda_N)$  Lewis and Aboav-Weaire laws are obeyed in these networks.

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# 1 Introduction

In the past three decades there has been a considerable interest in studying a class of non-equilibrium systems known collectively as *Cellular Structures*[1]. The geometrical and dynamical properties of these systems are best displayed in the familiar pattern formed by a soap froth confined between two transparent plates. Other examples include polycrystalline domains in metals, ceramics, magnetic domains, biological tissues and mono layers of fatty acids on surface of water. One can also mention other examples from material science, like cracking in glazes, fracture and dewetting of polymer films above their glass transition temperature [2]-[8]. Despite the diversity in systems in which cellular structures are formed, numerous experiments have shown that the long time statistical behavior of these systems are characterized by certain universal, system independent laws. This means that topological and geometrical constraints influence the properties of these networks in a very essential way. For example, the fact that energy is associated with the length of the edges of cells, immediately leads to the conclusion that in the two dimensional structures all the vertices are 3-valent. Combination of this result with the Euler character formula shows that the average number of sides per cell,  $\langle i \rangle$  is equal to 6:

$$\langle i \rangle = 6. \quad (1)$$

Similar considerations in 3-dimensions, where energy is associated with the area of faces of cells, proves that all vertices are 4-valent and that:

$$\langle f \rangle = \frac{12}{6 - \langle i \rangle}, \quad (2)$$

where  $\langle f \rangle$  and  $\langle i \rangle$  are the average number of faces per cell and the average number of edges per face respectively. Besides these properties which are a direct consequence of the Euler character formula, experiments have revealed a number of very general, properties among which the most important are : i) Von-Neumann [9] , ii) Lewis [10] and iii) Aboav-Weaire law [11]. Von-Neumann law refers to the rate of expansion of a single cell and is accounted for theoretically in a satisfactory way. It has also been generalized to curved surfaces where the curvature of the surfaces plays a role both in the dynamic of a single cell and in it's stability [12]. The Aboav-Weaire law which is statistical in nature and refers to the correlation of adjacent cells has been shown to hold in random 3-valent graphs, viewed as planar Feynman graphs of  $\phi^3$  theory and solved by techniques of matrix models [13]. The authors of [14] have investigated the statistical properties of two dimensional random cellular systems in term of their shell structures.

However there has been no explanation of the empirical Lewis law which states that the average area of cells has a linear relation with the number of cells, for large number of edges. However recently there have been some attempts [15] to derive both Lewis

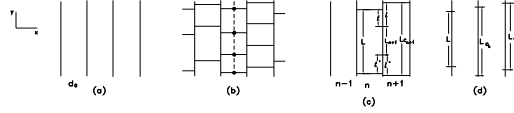


Figure 1: *2D Poisson network. (a) Parallel columns in the  $x$  direction with width  $d_0$ . (b) These columns are divided into cells by edges through the points of independent uniform or non-uniform poisson distributions. The structure in c is used for obtaining the probability density of cover lengths,  $L_c$ , for a given  $L$ . (d) Cover lengths of segment  $L$  in two distribution which partly overlap.*

and Aboav-weaire laws, from Poisson networks, i.e. networks based on a Poisson distribution of horizontal segments between a fixed set of parallel lines. In ref.[15] it was shown that in such networks both the Aboav-Weaire and the Lewis laws were obeyed. The analysis of ref. [15] was however restricted to the uniform density case, i.e: where the distance between the horizontal lines were all equal. In this paper we extend this analysis to the non-uniform case and add one more ingredient of randomness to the network, where the distribution of vertical lines is also probabilistic. The paper consists of two parts, the first part (section 2), where we rederive the results of [15] in a somewhat clearer way, and part two (section 3) where we extend these results to the non-uniform density case.

## 2 Poisson Networks With Uniform Density

The two dimensional Poisson networks are generated as follows (fig.1)[15]. One takes a family of parallel lines in the  $y$ -direction in the plane. The distance between lines does not affect the topological properties of the network, although it affects the geometry. This distance is taken as uniform and equal to  $d_0$ . Suppose there are  $N$  columns,  $C_1, C_2, \dots, C_N$ , in the network. The region between two successive columns is divided into cells. The division of each column  $C_\alpha$ , is based on a Poisson point distribution in the  $y$ -direction. An edge in the  $x$ -direction is taken through each Poisson point (P-point). In this way a 3-valent network is generated in the plane which, although different from the realistic cellular structures, is simple enough to be analyzed closely for the study of various statistical-topological properties of the network.

The 3-dimensional tetravalent network are generated by a similar process (fig.2). One takes an arbitrary 2-dimensional trivalent network(base network) on the  $xy$  plane

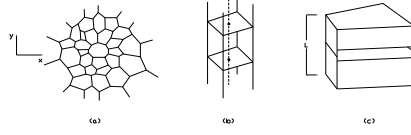


Figure 2: *Three dimensional Poisson network. (a) an arbitrary planar network. (b) The columnar structure of three dimensional Poisson network. The cell in c has  $i = 4$ ,  $F = 10$ , and height  $L$ .*

(fig.2.a) and takes vertical planes (parallel to the z-axis) through each edge which divides the 3-dimensional space into prismatic columns. In each column one considers a Poisson distribution of uniform density in the z-direction and divides the columns into cells by planes perpendicular to the z-axis through each P-point (fig.2.b). The height of a cell,  $L$ , is the distance between adjacent P-points in the associated distribution (fig.2.c). The average cell height is unity.

Consider one of the columns, say  $C_\alpha$ . The Poisson distribution of  $n$  points in a segment of length  $L$  is given by :

$$P(n) = \frac{L^n}{n!} e^{-L}. \quad (3)$$

Consider a particular set of columns  $C_\alpha, C_\beta, \dots, C_\gamma$ . The probability that in a given distance  $L$ , there are  $n_\alpha, n_\beta, \dots, n_\gamma$  segments respectively in columns  $C_\alpha, C_\beta, \dots, C_\gamma$ , is equal to

$$\Phi(L, n_\alpha, n_\beta, \dots, n_\gamma) \equiv P(n_\alpha)P(n_\beta) \dots P(n_\gamma). \quad (4)$$

Now consider a reference column which we denote by  $C_\alpha$ . The probability that there is a cell with height between  $L$  and  $L + dL$  in this column is  $e^{-L}dL$ . I denote the columns which are neighbors of this reference column by  $C_{\alpha_1}, C_{\alpha_2}, \dots, C_{\alpha_k}$ . The number of these neighboring columns depend on the dimensionality and type of the network. The joint probability that there is a cell of height between  $L$  and  $L + dL$  in the column  $C_\alpha$  such that its neighbors  $C_{\alpha_1}, C_{\alpha_2}, \dots, C_{\alpha_k}$  respectively have  $n_{\alpha_1}, n_{\alpha_2}, \dots, n_{\alpha_k}$  points in the interval  $L$  is :

$$\Psi_\alpha(L, n_{\alpha_1}, n_{\alpha_2}, \dots, n_{\alpha_k})dL = P(n_{\alpha_1})P(n_{\alpha_2}) \dots P(n_{\alpha_k})e^{-L}dL. \quad (5)$$

Clearly this distribution function depends on the individual values of the variables  $n_{\alpha_1}, n_{\alpha_2}, \dots, n_{\alpha_k}$ , however in the sequel another probability distribution will be useful. That is  $\Psi(L, I)dL$  where  $I$  is defined as  $I = n_{\alpha_1} + n_{\alpha_2} + \dots + n_{\alpha_k}$ . The joint probability

$\Psi(L, I)dL$ , is calculated as follows:

$$\Psi_{\alpha}(L, I)dL = \sum' \Psi_{\alpha}(L, n_{\alpha_1}, n_{\alpha_2}, \dots, n_{\alpha_k})dL = \frac{(kL)^I}{I!} e^{-(k+1)L} dL, \quad (6)$$

where  $\sum'$  means the sum over all  $n_{\alpha_i}$ 's subject to the constraint that their sum be equal to I. It's important to note that the distribution  $\Psi_{\alpha}(L, I)$  is obtained from a sum of distribution of the form  $\Psi_{\alpha}(L, n_{\alpha_1}, n_{\alpha_2}, \dots, n_{\alpha_k})$  and is not equal to any of them.(compare with eq.(9a) of[15]).

## 2.1 Statistical Properties of Poisson Networks

### 2.1.1 Two Dimensional Networks

In two dimensions, every column has two neighbors, hence in eq.(6),  $k = 2$ , and the number of edges of a cell (see fig.1) in a reference column is equal to :

$$i = I + 4. \quad (7)$$

Combining eqs.(6) and (7) gives the probability distribution  $g(L, i)$  of finding  $i$ -cells of height in the interval  $(L, L + dL)$ :

$$g(L, i)dL = \frac{(2L)^{(i-4)}}{(i-4)!} e^{-3L} dL. \quad (8)$$

Clearly this distribution function is normalized.

$$\int_0^{\infty} \sum_{i=4}^{\infty} g(L, i)dL = 1. \quad (9)$$

From eq.(8) one can obtain the average height of  $i$ -cells:

$$\langle L \rangle_i = \int_0^{\infty} L \frac{g(L, i)}{g(i)} dL, \quad (10)$$

where  $g(i)$  is the total probability distribution of  $i$  - cells which given by:

$$g(i) = \int_0^{\infty} g(L, i)dL = \frac{1}{3} \left(\frac{2}{3}\right)^{i-4}. \quad (11)$$

A simple computation now gives:

$$\langle L \rangle_i = \frac{i-3}{3}. \quad (12)$$

This equation, then can be used to calculate the average area of  $i$ -cells. For the Poisson networks (fig.1), where the width of all cells are equal to  $d_0$ , we find:

$$\langle A \rangle_i = d_0 \langle L \rangle_i = \frac{d_0(i-3)}{3}, \quad (13)$$

which as far as linear dependence on the number of edges is concerned, agrees with Lewis law. One can also define a generating function:

$$G(q) = \langle e^{iq} \rangle \equiv \sum_{i=4}^{\infty} g(i) e^{iq}, \quad (14)$$

a simple calculation shows that:

$$G(q) = \frac{e^{4q}}{3 - 2e^q}, \quad \text{or} \quad \ln G(q) = 4q - \ln(3 - 2e^q). \quad (15)$$

From which one obtains by successive differentiation various connected moments of the distribution:

$$\langle i \rangle = 6, \quad (16)$$

$$\langle i^2 \rangle - \langle i \rangle^2 = 6, \quad (17)$$

$$\langle i^3 \rangle - 3 \langle i^2 \rangle \langle i \rangle + 2 \langle i \rangle^3 = 30. \quad (18)$$

In the remaining part of this subsection we review briefly the basic steps of the analysis of ref.[15] for derivation of the Aboav - Weaire law. Consider an *i-cell*  $a$  of length  $L$  in a two dimensional Poisson network(fig.3). This cell is in column 0 and has two neighbors in this column, called  $a'$  and  $a''$ .

There are two adjacent neighbors 1 and 2, which respectively distribute  $n_1$  and  $n_2$  points inside the cell  $a$ . The number of sides of,  $a$ , is then equal to

$$i = n_1 + n_2 + 4. \quad (19)$$

The total number of sides of the cells in column 1 adjacent to  $a$  is:

$$J_1 = 4(n_1 + 1) + (V_1 + 1) + (V'_1 + 1) + V''_1, \quad (20)$$

where the meaning of the numbers  $V_1$ ,  $V'_1$  and  $V''_1$  are specified in fig.3. For obtaining the Aboav-Weaire law we use one of the three figures (fig.3a), (fig.3b) or (fig.3c) and results are equivalent. Clearly  $im_i$  which is the total number of sides of the cells adjacent to  $a$  is:

$$im_i = 12 + J_1 + J_2. \quad (21)$$

Where the definition of  $J_2$  is similar to that of  $J_1$  and the number 12 comes from the average of sides of  $a'$  and  $a''$ . For the average value of  $V_1$ ,  $V'_1$  and  $V''_1$  we use:

$$\langle V_1 + V'_1 \rangle = \frac{\langle L_{c_1} - L_a \rangle}{1} = \langle L_{c_1} - L_a \rangle, \quad (22)$$

$$\langle V''_1 \rangle = \frac{\langle L_{c_1} \rangle}{1} = \langle L_{c_1} - L_a \rangle + \langle L_a \rangle. \quad (23)$$

Where the segment  $L_{c_1}$  in column  $C_1$ , which is the smallest segment containing  $L_a$ , is called the covering length of  $L_a$  [15]. With the probability distribution found in [15]

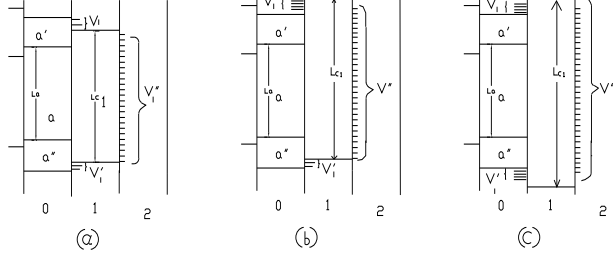


Figure 3: *These figures are used for obtaining Aboav - Weaire law in two and three dimensions.*

for  $L_c$  it is shown that the average  $\langle L_c - L_a \rangle = 2$ , in any arbitrary column, it then follows that:

$$\langle im_i \rangle = 24 + 4(n_1 + n_2) + \langle V_1 + V'_1 \rangle + \langle V_2 + V'_2 \rangle + \langle V''_1 \rangle + \langle V''_2 \rangle. \quad (24)$$

By substituting eqs.(10), (22), (23) and (24) one finds that:

$$\begin{aligned} \langle im_i \rangle &= 16 + 4i + 2 \langle L \rangle_i, \\ &= 16 + 4i + \frac{2(i-3)}{3} = 14 + \frac{14}{3}i. \end{aligned} \quad (25)$$

Combining eq.(25) with (12) one obtains

$$\langle im_i \rangle = 16 + 4i + \frac{2(i-3)}{3} = 14 + \frac{14}{3}i \quad (26)$$

This is in accord with Aboav-Weaire law, i.e:  $\langle im_i \rangle = c_1 + c_2 i$  with  $c_1$  and  $c_2$  which are 14 and  $\frac{14}{3}$  respectively.

### 2.1.2. Three-Dimensional Networks.

Consider a 3-dimensional network and a column whose base is an  $i$ -cell. In eq.(6), we should now equate  $k$  with  $i$  and, relate  $I$  with the number of faces as follows:

$$I + 2 + i = F. \quad (27)$$

The probability distribution of  $F$ -cells (cells with  $F$ -faces) of height between  $L$  and  $L + dL$ , is now obtained as:

$$g_i(L, F)dL = \frac{(iL)^{F-i-2}}{(F-i-2)!} e^{-(i+1)L} dL. \quad (28)$$

The total distribution of  $F$  - cells whose base is an  $i$  - cell is given by:

$$g_i(F) = \int_0^\infty g_i(L, F)dL = \frac{1}{(i+1)} \left( \frac{i}{i+1} \right)^{F-i-2}. \quad (29)$$

One can then obtain the average height of  $F$ -cells whose bases are  $i$ -cell.

$$\langle L \rangle_{i,F} = \int_0^\infty \frac{L g_i(L, F) dL}{g_i(F)} = \frac{F - i - 1}{(i + 1)}. \quad (30)$$

The average height of  $F$ -cells with any base is given by:

$$\langle L \rangle_F = \sum_{i=4}^{\infty} \langle L \rangle_{F,i} g(i), \quad (31)$$

where  $g(i)$  is the probability distribution of  $i$ -cells in the base. As an approximation one can assume that the base network is hexagonal and obtain:

$$\langle L \rangle_{6,F} = \langle L \rangle_F = \frac{F - 7}{7}. \quad (32)$$

So the average volume (Lewis law in three dimensions) of  $F$ -cells will than be:

$$\langle v \rangle_F = S \frac{F - 7}{7}, \quad (33)$$

where  $S$  is the average area of cells in the base. Also, as second example, we assume that the network of base is Poisson network with uniform density, which has explained in previous section(eq.(11)), so that eq.(31) will become:

$$\langle L \rangle_F = \frac{1}{3} \sum_{i=4}^{\infty} \langle L \rangle_{F,i} \left(\frac{2}{3}\right)^{i-4} = 0.155F - 1. \quad (34)$$

Therefore by making use of eq.(13) we can obtain the overage volume of  $F$  - cells in this model:

$$\langle v \rangle_F = \frac{d_0}{3} (i - 3)(0.155F - 1). \quad (35)$$

From the formula for  $g_i(F)$  various moments can be calculated:

$$\langle F \rangle_i = 2i + 2, \quad (36)$$

$$\langle F^2 \rangle_i = 5i^2 + 9i + 4, \quad (37)$$

and

$$\langle F \rangle = 2 \langle i \rangle + 2 = 14, \quad (38)$$

$$\langle F^2 \rangle = 5 \langle i^2 \rangle + 9 \langle i \rangle + 4 = 268. \quad (39)$$

### 3 Poisson Networks With Non-Uniform Density

A little modification of the previous formulas is necessary to study the properties of a network where the density of points (the average distance between points to be denoted by  $\lambda$ , in the following) in different columns are different. Instead of eqs.(3) and (4) we have:

$$P_\alpha(n) = \left(\frac{L}{\lambda_\alpha}\right)^n \frac{1}{n!} e^{-\frac{L}{\lambda_\alpha}}, \quad (40)$$



$$\begin{aligned}
\Phi_0(L, n_1, n_2, \dots, n_k) &= P_1(n_1)P_2(n_2) \dots P_k(n_k), \\
&= \prod_{\alpha=1}^k \left(\frac{L}{\lambda_\alpha}\right)^{n_\alpha} \frac{1}{n_\alpha!} e^{-\frac{L}{\lambda_\alpha}},
\end{aligned} \tag{41}$$

respectively. By defining  $\Psi(L, I)dL$  as before, we obtain:

$$\begin{aligned}
\Psi_0(L, I) &= \sum_{n_1+n_2+\dots+n_k=I}^I \Phi(L, n_1, n_2, \dots, n_k) \frac{1}{\lambda_0} e^{-\frac{L}{\lambda_0}} dL, \\
&= \frac{L^I (\xi_0 - \frac{1}{\lambda_0})^I}{\lambda_0 I!} e^{-L\xi_0},
\end{aligned} \tag{42}$$

where  $\xi_0 = \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_k}$ . Now we study another probability density. We consider a cell of length  $L$  in which randomly located in the  $C_n$ th column ( $L$  is Poisson segment in  $C_n$ th column). The cover length,  $L_{c_{n+1}}$ , of  $L$  is defined as the smallest length in  $C_{n+1}$ th column such as it is contain length  $L$  (i.e.  $L_{c_{n+1}} \geq L$ ) (fig(1c)). Here, we wish to obtain the probability density of the cover length,  $L_{c_{n+1}}$ , whereas the main distribution in the columns of network is inhomogenous Poisson. So this new probability destiny must satisfy the following conditions:

1) In the constant length  $L_{n+1}$  of  $L$ , there are  $m$  points of  $C_{n+1}$ th non-uniform Poisson distribution, i.e.

$$P_{n+1}(m, L_{n+1}) = \left(\frac{L_{n+1}}{\lambda_{n+1}}\right)^m \frac{1}{m!} e^{-\frac{L_{n+1}}{\lambda_{n+1}}}. \tag{43}$$

2) There are segments  $l$  and  $l'$  in  $C_{n+1}$ th non-uniform density

$$P_{(n+1)}(l) = \frac{1}{\lambda_{n+1}} e^{-\frac{l}{\lambda_{n+1}}}, \tag{44}$$

$$P_{(n+1)}(l') = \frac{1}{\lambda_{n+1}} e^{-\frac{l'}{\lambda_{n+1}}}, \tag{45}$$

and

$$L_{c_{n+1}} = L_{n+1} + l + l', \tag{46}$$

$$L = L_{n+1} + t + t', \tag{47}$$

$$t \leq l \leq L_{c_{n+1}} - L + t. \tag{48}$$

3)

$$\int_L^\infty \mathcal{P}_{n+1}(L_{c_{n+1}}, L) dL_{c_{n+1}} = 1. \tag{49}$$

where  $\mathcal{P}_{n+1}(L_{c_{n+1}}, L)$  is the non-uniform probability density of cover length  $L_{c_{n+1}}$ . Therefore, the probability density of cover length in  $C_{n+1}$ th non-uniform Poisson distribution with above conditions is:

$$\mathcal{P}_{n+1}(L_{c_{n+1}}, L, m) =$$

$$Q \int_t^{L_{c_{n+1}}-L-t} \int P_{n+1}(l) P_{n+1}(l') P_{n+1}(m, L_{n+1}) \delta(L_{c_{n+1}} - L_{n+1} - l - l') dl dl', \tag{50}$$

where Q is a normalization factor. One can find probability density of cover length for any m:

$$\mathcal{P}_{n+1}(L_{c_{n+1}}, L) = \sum_{m=0}^{\infty} \mathcal{P}_{n+1}(L_{c_{n+1}}, L, m) = \frac{1}{\lambda_{n+1}^2} (L_{c_{n+1}} - L) e^{-\frac{1}{\lambda_{n+1}}(L_{c_{n+1}} - L)}. \quad (51)$$

We see that, this distribution is independent of  $L_{n+1}$  and  $t$ . The average value of  $(L_{c_{n+1}} - L)$  for fixed L in column  $C_{n+1}$  is:

$$\langle (L_{c_{n+1}} - L) \rangle = 2\lambda_{n+1}. \quad (52)$$

Note that, it is independent of L. In the case which we study the 3D poisson network, for a given segment  $L$ , there are two cover lengths,  $L_{c_k}$  and  $L_{c_j}$  in two independent poisson distribution, in which are neighbour with each other (fig1.d). It is possible that these two cover lengths partly overlap. The probability density of the non-overlap length  $|L_{c_k} - L_{c_j}|$  is  $\frac{1}{\lambda_k} \exp(-\frac{L_{c_k} - L_{c_j}}{\lambda_k})$ , for  $C_k$ th column. The average of the non-overlap length in  $C_k$ th column is:

$$\langle |L_{c_k} - L_{c_j}| \rangle = \lambda_k. \quad (53)$$

### 3.1 Statistical Properties of Non-Uniform Density

#### 3.1.1 Two Dimensional Networks

In a two dimensional network every column say  $C_n$  has two neighbours,  $C_{n-1}$  and  $C_{n+1}$ . The number of sides of  $i$ -cell in column  $C_n$ , is  $i = 4 + I$ , where  $I = n_{n-1} + n_{n+1}$  and  $n_{n-1} + n_{n+1}$  are the number of vertices contributed by the cells in adjacent columns. Then for the distribution of  $i$ -cells of length between  $(L$  and  $L + dL)$  in column  $C_n$ , we obtain:

$$g_n(L, i) = \frac{L^{i-4} (\xi_n - \frac{1}{\lambda_n})^{i-4}}{\lambda_n (i-4)!} e^{-L\xi_n}, \quad (54)$$

where  $\xi_n = \frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n} + \frac{1}{\lambda_{n+1}}$ . Hence, the distribution of  $i$ -cells with any arbitrary length in column  $C_n$  is:

$$g_n(i) = \int_0^{\infty} g_n(L, i) dL = \frac{1}{\lambda_n \xi_n} \left(1 - \frac{1}{\lambda_n \xi_n}\right)^{i-4}, \quad (55)$$

clearly this distribution function is normalized. From eq.(48) and (49) we also obtain the average height of  $i$ -cells in column  $C_n$ :

$$\langle L \rangle_{n,i} = \int_0^{\infty} \frac{L g_n(L, i) dL}{g_n(i)} = \frac{i-3}{\xi_n}. \quad (56)$$

So that for the Poisson networks with non-uniform density in one direction ( $y$  direction)(fig1) Lewis law (average area of  $i$ -cells) in column  $C_n$  is:

$$\langle A \rangle_{n,i} = \frac{d_0(i-3)}{\xi_n}. \quad (57)$$

One can now evaluate various moments of the distribution  $g_n(i)$ . In all the following calculations, we use the following formulas for geometric sum:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad (58)$$

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}, \quad (59)$$

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(1+x)}{(1-x)^3}. \quad (60)$$

A simple calculation shows that:

$$\langle i \rangle_n = \sum_{i=4}^{\infty} i g_n(i) = 3 + \lambda_n \xi_n = 4 + \lambda_n \left( \frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_{n+1}} \right), \quad (61)$$

$$\langle i^2 \rangle_n = \sum_{i=4}^{\infty} i^2 g_n(i) = 2(\lambda_n \xi_n)^2 + 5\lambda_n \xi_n + 9, \quad (62)$$

where  $\langle i \rangle_n$  is the average number of  $i$ -cells in column  $C_n$ . The variance turns out to be equal to:

$$\langle i^2 \rangle_n - \langle i \rangle_n^2 = (\lambda_n \xi_n)(\lambda_n \xi_n - 1). \quad (63)$$

In order to find the moments  $\langle i \rangle$  and  $\langle i^2 \rangle$  in the whole lattice and not in a particular column  $C_n$ , we replace the average over the cells in a network, by an average over the densities in an ensemble of networks. Thus from eqs.(57), (61) and (??) we have :

$$\ll A \gg_i = d_0(i-3) \langle \frac{1}{\xi_n} \rangle, \quad (64)$$

$$\ll i \gg = 4 + \langle \lambda_n \left( \frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_{n+1}} \right) \rangle, \quad (65)$$

$$\ll i^2 \gg = 2 \langle (\lambda_n \xi_n)^2 \rangle + 5 \langle \lambda_n \xi_n \rangle + 9. \quad (66)$$

Where the averages on the right hand side are performed with a suitable distribution

$$P(\lambda_1, \lambda_2, \dots, \lambda_N).$$

For simplicity in the following we assume that this distribution, is a symmetric distribution, i.e:

$$P(\lambda_1, \lambda_2, \dots, \lambda_N) = P(\lambda_{1'}, \lambda_{2'}, \dots, \lambda_{N'}), \quad (67)$$

where the primed indices are a permutation of the unprimed ones. Clearly this minor restriction allows our conclusions to be valid for a very large class of probability distributions. Finally, we consider Aboav-Weaire law for non-uniform Poisson distribution. By using of eqs.(22),(23) and (52), we obtain:

$$\langle V_1 + V'_1 \rangle_{n+1} = \frac{\langle L_{C_{n+1}} - L_a \rangle_{n+1}}{\lambda_{n+1}} = 2, \quad (68)$$

$$\langle V_1'' \rangle_{n+1} = \frac{\langle L_{c_{n+1}} \rangle_{n+1}}{\lambda_{n+1}} + \frac{\langle L_a \rangle_{n+1}}{\lambda_{n+1}} = 2 + \frac{1}{\lambda_{n+1}} \langle L_a \rangle_{n+1}. \quad (69)$$

Combining eqs.(24), (68) and (69), we have:

$$\langle im_i \rangle = 16 + 4i + \frac{\langle L_a \rangle_{(n+1),i}}{\lambda_{n+1}} + \frac{1}{\lambda_{n-1}} \langle L_a \rangle_{(n-1),i}. \quad (70)$$

By using of eqs.(57) and (70), obtain:

$$\langle im_i \rangle = 16 + 4i + (i-3) \left( \frac{1}{\lambda_{n+1}\xi_{n+1}} + \frac{1}{\lambda_{n-1}\xi_{n-1}} \right). \quad (71)$$

At last we have:

$$\ll im_i \gg = 16 + 4i + (i-3) \left( \frac{1}{\lambda_{n+1}\xi_{n+1}} + \frac{1}{\lambda_{n-1}\xi_{n-1}} \right). \quad (72)$$

Where the last averaging are performed by a given distribution in eq.(67). If we use of uniform distribution we have  $\lambda_{n+1}\xi_{n+1} = \lambda_{n-1}\xi_{n-1} = 3$ , so that:

$$\ll im_i \gg = 14 + \frac{14}{3}i, \quad (73)$$

we see that this result is the same in eq.(26)

### 3.1.2 Three-Dimensional Networks

Consider a 3-dimensional laminated Poisson network, based on an arbitrary 2-dimensional network, and a particular  $i$ -cell is the base. I label the column with this base by  $C_{i,0}$ . The number of faces in a cell in the column above this  $i$ -cell is

$$F = i + 2 + J, \quad (74)$$

where  $J = n_1 + n_2 + \dots + n_i$ , and  $n_k$  is the number of additional lateral faces which results from the cells in the adjacent  $k$ -column (fig.3). Let  $f_i(J)$  be the fraction of the  $F$ -cells in column  $C_{i,0}$ . In formula (41), one should now replace  $k$  by  $i$ , and  $I$  by  $J$  to obtain:

$$\Psi_0(L, F, i) = \frac{L^{F-i-2} \left( \xi_0 - \frac{1}{\lambda_0} \right)^{F-i-2}}{\lambda_0 (F-i-2)!} e^{-L\xi_0}, \quad (75)$$

where  $\xi_0 = \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_i}$ . Here  $\lambda_0$  is the density of P-point distribution in the column  $C_{i,0}$  and  $\lambda_\alpha$  ( $\alpha = 1, 2, \dots, i$ ) are the density of P-point distribution in the adjacent columns.  $\Psi_0(L, F, i)dL$  is the probability of finding an  $F$ -cell in column  $C_{i,0}$ , whose height is between  $L$  and  $L + dL$ . Integrating over  $L$ , one obtains

$$\Psi_0(F, i) = \frac{1}{\lambda_0 \xi_0} \left( 1 - \frac{1}{\lambda_0 \xi_0} \right)^{F-i-2}, \quad (76)$$

this is the probability of finding an  $F$ -cell in column  $C_{i,0}$ . The average height of  $F$ -cells with  $i$ -cell base in column  $C_0$  is:

$$\langle L \rangle_{0,i,F} = \int_0^\infty \frac{L \Psi_0(F, L, i) dL}{\Psi_0(F, i)} = \frac{F-i-1}{\xi_0}, \quad (77)$$

hence, the average height of  $F$  - cells with any base in column  $C_0$

$$\langle L \rangle_{0,F} = \sum_{i=4}^{\infty} \langle L \rangle_{0,i,F} g_0(i). \quad (78)$$

Here, we can calculate eq.(78) for three different base networks. Assume that the network of base is:

1) hexagonal

$$\langle L \rangle_{0,F} = \frac{F-7}{\xi_0}, \quad (79)$$

2) uniform Poisson network

$$\langle L \rangle_{0,F} = \frac{1}{3} \sum_{i=4}^{\infty} \left( \frac{F-i-1}{\xi_0} \right) \left( \frac{2}{3} \right)^{i-4}, \quad (80)$$

3) non-uniform Poisson network

$$\langle L \rangle_{0,F} = \sum_{i=4}^{\infty} \left( \frac{F-i-1}{\xi_0} \right) g_0(i), \quad (81)$$

note that the non-uniform Poisson distribution in horizontal columns (2D Poisson network) and vertical columns are independent. Therefore we can easily calculate the final averaging in eq.(74). One can now find various moments of this distribution:

$$\langle F \rangle_{i,0} = \lambda_0 \xi_0 + i + 1, \quad (82)$$

$$\langle F^2 \rangle_{i,0} = (i+1)^2 + (2i+1)(\lambda_0 \xi_0) + 2(\lambda_0 \xi_0)^2. \quad (83)$$

By the same averaging procedure as in the two dimensional case one can obtain:

$$\ll L \gg_{F,i} = (F-i-1) \langle \frac{1}{\xi_0} \rangle, \quad (84)$$

$$\ll F \gg_i = \langle \lambda_0 \xi_0 \rangle + i + 1, \quad (85)$$

$$\ll F^2 \gg_i = (i+1)^2 + (2i+1) \langle \lambda_0 \xi_0 \rangle + 2 \langle (\lambda_0 \xi_0)^2 \rangle. \quad (86)$$

The later averages are performed over the density distributions eq.(67). Note that for the distribution with uniform density in vertical columns one has:  $\lambda_0 \xi_0 = i+1$  and the eqs.(84), (85) and (86) reduce to:

$$\ll L \gg_{i,F} = \frac{F-i-1}{i+1}, \quad (87)$$

$$\ll F \gg_i = 2i+2, \quad (88)$$

$$\ll F^2 \gg_i = (i+1)(5i+4), \quad (89)$$

which are in agreement with equations (30), (36), (38).

In this section, we derive the 3D Aboav - Weaire law in non-uniform Poisson network. Consider a 3D cell  $a$  with  $F$  faces, height  $L_a$  and a base with  $i$  edges in reference column,  $C_0$ . We want find the total number of faces in cells adjacent to  $a$  with fixed  $F$

and  $i$ . This quantity is  $(Fm_F)_{0,i}$ . The number lateral walls in column  $k$  is  $m_i$ . So that the total number of faces, which obtain from P-points within  $L_a$  in first - neighbour columns to  $a$  is:

$$J = \sum_{k=1}^i J_k = \sum_{k=1}^i [(m_i + 2)(n_k + 1) + (V_k + 1) + (V'_k + 1)], \quad (90)$$

and, the total number of faces, which obtain from P-point within second- neighbour columns to  $a$  is:

$$J' = \sum_{k=1}^{im_i - 3i} V'_k = (im_i - 3i) \langle V'_k \rangle, \quad (91)$$

where,  $(im_i - 3i)$  is the number of vertical walls of the adjacent cells that contact neighbours, and also there are another faces which obtain from P-points in pairs of adjacent column  $k, j$  that are adjacent to each other. So the total number of extra faces of such adjacenceis is

$$J'' = 2 \sum_{k=1}^i [(n_k + 1) + \frac{\langle |L_{c_k} - L_{c_j}| \rangle}{\lambda_k}], \quad (92)$$

where the number 2 is due to two distributions are overlap in each vertical wall. Clearly one can obtain  $(Fm_F)_{0,i}$  as

$$\langle Fm_F \rangle_{0,i} = 28 + J + J' + J''. \quad (93)$$

Hence the number 28 comes from the average of faces of two cells adjacent to  $a$  in the same reference column,  $C_0$ . By substituting eqs. (90), (91), (92) in (93), and using of (52), (53), (68), (69) and (77), we have

$$\langle Fm_F \rangle_{0,i} = m_i \left[ \left(1 + \frac{i}{\xi_0}\right) F + \left(2 - \frac{(i+1)}{\xi_0}\right) i - 2 \right] + \left[4 - \frac{3i}{\xi_0}\right] F + \frac{3i(i+1)}{\xi_0} + 20, \quad (94)$$

and at last, we have

$$\begin{aligned} \ll Fm_F \gg_i &= m_i \{ [i + i < \frac{1}{\xi_0}] F + [2 - (i+1) < \frac{1}{\xi_0}] i - 2 \} \\ &+ \{ 4 - 3i < \frac{1}{\xi_0} \} F + 3i(i+1) < \frac{1}{\xi_0} > + 20. \end{aligned} \quad (95)$$

Hence the double average are performed by a symmetric distribution such as (70). It is remarkable that these averaging are performed with respect to those distributions in vertical columns (for example  $z$  direction), and are independent of distributions in base network. So that if we use of non-uniform poisson network as base network, we have two averaging, which are independent.

## 4 Conclusion

My main conclusion is that Lewis and Aboav- Weaire laws apply to Poisson networks. I have shown that by more randomizing these networks; (i.e. in the sense of random density distribution in different columns) one can still obtain the above mentioned laws. In fact the most features of these networks have in common with real cellular structures. But the only property of real systems in which all the angles around any vertex are equal is obviously absent in these networks. So we see that these model satisfy Lewis and Aboav-Weaire laws with the coefficients which are slightly different from the results obtained in [1,9]. Therefore despite this modification the main features of this model is still unaffected and one can derive the statistical and topological properties of random cellular structures from this modified model.

## References

- [1] For a review see : J. Stavans ; " The Evolution of Cellular Structures " Rep. Prog. Phys. **56**(1993) 733-789, and references therein.
- [2] J. Stavans, E. Domany, D. mukamel; Europhysics Letter **15** (1991) 479-84
- [3] B. Levitan, E. Domany; Int. J. Mod. Phys **B 10** (1996) 3765-3805
- [4] B. Levitan; Phys. Rev. **E**,Vol.**55** (1997) 1205-8
- [5] W. Korneta, S. K. Mentiratta, J. Menteiro; Phys. Rev. **E**,Vol.**55** (1997) 1205-8
- [6] N. Rivier, M. A. Peshkin, B. Dubertret; J. Phys. A : Math. Gen. **31** (1985) 879-900
- [7] C. G. Le; J. Phys. A : Math. Gen. **24** (1991)4655-75
- [8] C. G. Le, P. V. Parfait, R. Delannay; Euro. Phys. J. **B4** (1998) 499-511
- [9] J. V. Neumann " Metal Interfaces " ( Cleveland, OH: American Society for Metals) p. 108
- [10] F. T. Lewis; Anat. Rec. **38** (1928) 241
- [11] D. A. Aboav; Metallography **3** (1970) 383-90 ; Metallography **13** (1980) 43-58
- [12] J. E. Avron, D. Levine; Phys. Rev. Letts. **69** (1992) 208
- [13] C. Godreche, I. Kostov, I. Yekuteli; Phys. Rev. Letts. **69** (1992) 2674
- [14] T. Aste, K. Y. Szeto, W. Y. Tam; Phys. Rev. **E**,Vol.**54** (1996) 5482-92
- [15] M. A. Fortes; J. Phys. A : Math. Gen. **28** (1995)1055-1068
- [16] M. A. Fortes; Phil. Mag. Lett. **68** (1993) 69-71